Study of the sound radiation of a rectangular plate resting on a winkler elastic foundation

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Abstract

In this work, the effect of a Winkler elastic foundation on the sound radiation of a rectangular plate is both analytically and numerically studied using the Resistance Radiation Matrix method. An exact solution of the damped natural frequencies of the free and forced vibration of the plate with different boundary conditions is presented. The effect of the stiffness of the elastic foundation on the sound power radiated by the plate is also investigated. It is concluded that the damped natural frequencies for both SS-C-SS-C and for a SS-SS-SS-SS plate, depend upon the stiffness of the elastic foundation. However, the mode shapes of the plate do not depend on this parameter. On the other hand, it is shown that, in the case of forced vibration, the sound power level of the plate has an inverse relationship with the stiffness of the Winkler elastic foundation. This feature is correctly described by the proposed mathematical model.

Keywords: Sound radiation, rectangular plate, elastic foundation
1 Introduction

The study of the sound radiation of plates resting on an elastic foundation has great importance in industry and it is has been an active research subject for many years. In this sense, many structural components in aerospace, civil, mechanical, and marine engineering are supported on an elastic medium [1]. Some previous works by Ascione and Grimaldi [2], Salari et al. [3], Zheng and Zhou [4], Leissa [5], Ghosh [6], Wang [7], and Bhaskara and Kameswara [8] have analyzed the vibration characteristics of plates resting on a Winkler elastic foundation. Other studies presented by Arenas and Crocker [9], Rdzanek et al. [10][11], Arenas [12][13] and Arenas et al. [14] have presented methods for estimating the sound power radiated from vibrating plates. However, there is a relative scarcity of information in the literature for the relationship between the sound power radiated from a vibrating plate resting on an elastic foundation and the stiffness of the elastic foundation. Thus, this paper presents a method to estimate the effect of an elastic foundation on the sound radiation of a rectangular plate.

The method is defined as a "lumped parameter model" [15][16] and it is based on a singular value decomposition (SVD) of a matrix which is computed through a division of the vibrating surface into a finite number of small circular piston sources [17][18][19]. The calculation of the sound radiation is based just on surface velocity information and a direct numerical evaluation of the radiation resistance matrix of the plate.

2 Formulation of the system

2.1 Free vibration

Consider a thin rectangular plate supported on a Winkler foundation. The following fourth-order differential equation describes the free flexural vibrations of a thin rectangular uniform plate [1]

\[
D \nabla^4 W(x,y,t) + \rho h \frac{\partial^2 W(x,y,t)}{\partial t^2} + K_f W(x,y,t) = 0
\]

(1)

Where \( D = \frac{E h^3}{12(1-\nu^2)} \) is the flexural rigidity of the plate, \( K_f \) is the stiffness of the Winkler foundation, and \( h, \rho, E, \nu \) are the plate's thickness, density, Young’s modulus, and Poisson’s ratio, respectively.

The displacement \( W(x,y,t) \) in eq. (1) can be written as a combination of spatial- and time-dependent components as \( W(x,y,t) = W(x,y) \sin(\omega t + \phi) \), then eq. (1) is written as

\[
D \nabla^4 W(x,y) + \rho h W(x,y) \omega^2 + K_f W(x,y) = 0
\]

(2)

Where \( \omega \) is the natural frequency of vibration of the plate. The general solution of eq. (2) takes the following form
\[(\nabla^4 - k^4) = 0\]  \hspace{1cm} (3)

where \(k^4 = \frac{(\rho_s\omega^2 - K_f)}{D}\) and \(\rho_s = \rho h\) is the surface density of the plate. The differential operator \((\nabla^4 - k^4)\) may be expressed as the product \((\nabla^2 + k^2)(\nabla^2 - k^2)\) and the solution of eq. (3) can be separated into two parts

\[(\nabla^2 - k^2)W(x, y) = 0 \text{ and } (\nabla^2 + k^2)W(x, y) = 0\]  \hspace{1cm} (4)

We analyzed two boundary conditions: the plate simply supported-clamped-simply supported-clamped and the plate simply supported on its 4 edges.

### 2.1.1 Plate simply supported on its 4 edges (SS-SS-SS-SS)

If \(k^2 > \alpha^2\), the solution of eq. (3) is [20]

\[W(x, y) = (A_m \sin(\sqrt{k^2 - \alpha^2}y) + B_m \cos(\sqrt{k^2 - \alpha^2}y) + C_m \sinh(\sqrt{k^2 + \alpha^2}y) + D_m \cosh(\sqrt{k^2 + \alpha^2}y)) \sin(ax)\]  \hspace{1cm} (5)

where \(A_m, B_m, C_m,\) and \(D_m\) are mode shapes parameters which can be determined from the boundary conditions. The boundary conditions for this case are [21]:

\[W = \frac{\partial^2 W}{\partial x^2} = \frac{\partial^2 W}{\partial y^2} = 0 \text{ over } x = 0, a \text{ and } y = 0, b\]  \hspace{1cm} (6)

Substituting eq. (5) into eq. (6), results in \(B_m = D_m = 0\) and we obtain that

\[\begin{bmatrix} \sin(\phi_1) & \sinh(\phi_2) \\ -\phi_1^2 \sin(\phi_1) & \phi_2^2 \sinh(\phi_2) \end{bmatrix} \begin{bmatrix} A_m \\ C_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\]  \hspace{1cm} (7)

where \(\phi_1 \equiv \left(b/a\right)\sqrt{\lambda^2 - m^2\pi^2}, \phi_2 \equiv \left(b/a\right)\sqrt{\lambda^2 + m^2\pi^2}\) and \(\lambda^2\) is the dimensionless frequency parameter given by \(\lambda^2 = (ka)^2 = a^2 \sqrt{\frac{\rho_s\omega^2 - K_f}{D}}\).

Setting the determinant of the coefficient matrix equal to zero in eq. (7), we derive the following equations for the natural damped frequencies (eq. (8)) and damped eigenfunctions (eq. (9)) of the plate as
\[ \omega_{mn} = \sqrt{\frac{D}{\rho_s}} \left[ \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \right]^2 + \frac{K_f}{\rho_s} \]  

(8)

\[ W(x, y) = A_{mn} \sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) \]  

(9)

where \( A_{mn} \) is the complex velocity amplitude of mode \((m, n)\) and may be determined from the initial conditions. It is important to note that damped eigenfunctions do not depend upon the stiffness of the Winkler foundation.

### 2.1.2 Plate simply supported-clamped-simply supported-clamped (SS-C-SS-C)

For two opposite edges simply supported, if \( k^2 > \alpha^2 \), the solution of eq. (3) is given by eq. (5) [22]. Now, the boundary conditions for this case are [23]

\[ W = \frac{\partial W}{\partial x} = \frac{\partial W}{\partial y} = 0 \text{ over } x = 0, a \text{ and } y = 0, b \]  

(10)

We use the same methodology of Section 2.1.1, and we derive the following equations for the natural damped frequencies (eq. (11)) and damped eigenfunctions (eq. (12)) of the plate as

\[ \omega_{mn} = \sqrt{\frac{D}{\rho_s}} \left[ \pi^2 \left( \frac{m^2}{a^2} + \frac{4n^2}{b^2} \right) \right]^2 + \frac{K_f}{\rho_s} \]  

(11)

\[ W(x, y) = A_{mn} \sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{2n\pi}{b} y \right) \]  

(12)

Again we can notice that damped eigenfunctions do not depend upon the stiffness of the Winkler foundation.

### 2.2 Forced vibration

In this section we studied the same plate of Section 2.1 but now the plate is excited by a force \( f(x, y, t) \). Eq. (13) describes forced flexural vibrations of the plate.
\[ D \nabla^4 W(x, y, t) + \rho h \frac{\partial^2 W(x, y, t)}{\partial t^2} + K_f W(x, y, t) = f(x, y, t) \]  

(13)

If the plate is excited by an arbitrary harmonic point force of amplitude \( F = F_0 \), fixed at \( x = x_0 \) and \( y = y_0 \), we can write that

\[ f(x, y, t) = F_0 e^{j\omega t} \delta(x - x_0) \delta(y - y_0) \]  

(14)

where \( \delta(\cdot) \) is the delta function. Substitution of eq. (14) into eq. (13) and assuming harmonic solutions \( W(x, y, t) = W(x, y)e^{j\omega t} \), yields the following equation

\[ \nabla^4 W(x, y) + \frac{\rho_s \omega_{mn}^2 - K_f}{D} + W(x, y) = \frac{F_0}{D} \delta(x - x_0) \delta(y - y_0) \]  

(15)

On the other hand, for a harmonically excited simply supported plate, the displacement at a location \((x, y)\) of the plate is given by Fuller et al. [24] as

\[ W(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K_{mn} \sin \left( \frac{m\pi}{l_x} x \right) \sin \left( \frac{n\pi}{l_y} y \right) \]  

(16)

where \( K_{mn} \) is the amplitude of the \((m, n)\) mode and \( l_x \) and \( l_y \) are the dimensions of the plate in the \( x \) and \( y \) directions. The modal amplitude \( K_{mn} \) can be written as \( K_{mn} = A_{mn} F_{mn} \).

where \( A_{mn} \) is at complex resonance parameter defined as \( A_{mn} = \frac{1}{\rho h(\omega_{mn}^2 - \omega^2 + j\omega D_{mn})} \), \( \omega_{mn} \) is the damped natural frequency of mode \((m, n)\), \( D_{mn} \) is the damping of mode \((m, n)\) given by \( D_{mn} = 2\zeta \omega_{mn} \) (where \( \zeta \) is the damping ratio), and \( \omega \) is the frequency of the excitation point force.

If a point force at a position \((x_0, y_0)\) is applied on the plate, the modal force \( F_{mn} \) for the simply supported plate is

\[ F_{mn} = 4 \frac{F_0}{l_x l_y} \sin \left( \frac{m\pi}{l_x} x_0 \right) \sin \left( \frac{n\pi}{l_y} y_0 \right) \]  

(17)

where \( F_0 \) is the complex force amplitude. Substituting eq. (17) into eq. (15), and considering \( l_x = a \) and \( l_y = b \), we obtain
\[ W(x, y) = 4 \frac{F_0}{\rho_s ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) \sin \left( \frac{m\pi}{a} x_0 \right) \sin \left( \frac{n\pi}{b} y_0 \right)}{\omega_{mn}^2 - \omega^2} \]  

(18)

On the other hand, the normal velocity at any location \((x, y)\) on the surface of the plate is [13]

\[ V(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \sin(k_m x) \sin(k_n y) \]  

(19)

where \(V_{mn}\) is the complex velocity amplitude of mode \((m, n)\), \(k_m = \frac{m\pi}{a}\) and \(k_n = \frac{n\pi}{b}\). If the summations in eq. (21) are truncated to a finite number of modes \(M \times N\), we obtain [13]

\[ V(x, y) = \alpha^T(x)A\beta(y) \]  

(20)

where \(\alpha(x)\) is an \(M \times 1\) vector of elements \(\alpha_m = \sin(k_m x)\), \(\beta(y)\) is an \(N \times 1\) vector of elements \(\beta_n = \sin(k_n y)\), and \(A\) is an \(M \times N\) matrix of complex velocity amplitudes. If the plate is excited by a harmonic point force concentrated at point \((x_0, y_0)\), the elements of matrix \(A\) are [13]

\[ A_{mn} = 4 \frac{j\omega F}{M} \frac{\alpha_m(x_0)\beta_n(y_0)}{\omega_{mn}^2 (1 + j\eta) - \omega^2} \]  

(21)

where \(F\) is the amplitude of the force, \(M\) is the total mass of the plate, and \(\eta\) is the total damping loss factor of the plate. For a rectangular plate divided into \(N\) equal small elements, the space-averaged mean square normal vibration velocity amplitude \(\langle V^2 \rangle\) can be estimated by [13]

\[ \langle V^2 \rangle = \frac{N}{2(ab)^2} \sum_{j=1}^{N} |u_j|^2 = \frac{N}{2(ab)^2} u^H u \]  

(22)

Where \(u\) is the \(N \times 1\) complex volume velocity vector, and \(H\) denotes the Hermitian. Now, if we write the elements of \(u\) as \(u_i = |u_i|e^{j\phi_i}\), the time-averaged total sound power is [9]

\[ \bar{\Pi}_{\text{rad}}(\omega) = \frac{1}{2} \sum_{i=1}^{N} R_{ii} |u_i|^2 + \frac{1}{2} \sum_{i=1}^{N} \sum_{k \neq i}^{N} R_{ik} |u_i||u_k| \cos(\phi_k - \phi_i) \]  

(23)
where $R_{ii}$ is the self-resistance part of the acoustic radiation resistance matrix and $R_{ik}$ is the cross-resistance part of the acoustic radiation resistance matrix. The values for $R_{ii}$ are given by [25]

$$R_{ii} = \rho c S_i \left[ 1 - \frac{J_1(2k a_i)}{k a_i} \right]$$  \hspace{1cm} (24)

where $c$ is the speed of sound in the fluid, $J_1$ is the first-order Bessel function, $S_i$ is the surface of the equivalent piston, and $a_i = \sqrt{S_i/\pi}$ is the radius of the piston.

The values for the cross-resistance elements can be approximately calculated by [26]

$$R_{ik} \approx \frac{2\rho k^2 S_i S_k}{\pi} \left[ \frac{J_1(k a_i)}{k a_i} \frac{J_1(k a_k)}{k a_k} \right] \sin(k r_{ik})$$  \hspace{1cm} (25)

where $S_k$ is the surface of the equivalent piston, $a_k = \sqrt{S_k/\pi}$ is the radius of the piston, and $r_{ik}$ is the distance between the center points of each piston.

### 3 Numerical results

Numerical simulations were carried out for the calculation of the sound power level of the plate. We consider a rectangular aluminium plate ($E = 7.1 \times 10^10$ N/m$^2$, $\rho = 2700$ kg/m$^3$, $\nu = 0.33$) with a surface $0.5 \times 0.6$ m$^2$ and thickness $h = 3$ mm. In addition, different values for both damping loss factor of the plate $\eta$ and stiffness of the Winkler foundation $K_f$, were used. We also studied the effects of the damping loss factor of the Winkler foundation $\eta_f$ and the damping loss factor of the plate $\eta_p$ on the sound power level of the plate. Thus, the Young’s modulus of the plate is written as a complex stiffness $E = E_0(1 + j\eta_p)$, where $E_0$ it’s the Young’s modulus of the plate’s material, and the stiffness of the Winkler foundation it’s given by $K = K_f(1 + j\eta_f)$. The infinite series are truncated to a finite number (100). Moreover, to give an indication of the effects of increasing the internal damping of the plate and the elastic foundation, values of damping loss factors between 0 and 0.1 were chosen. Finally, values of stiffness between 0.55 and 24.5 kgf/mm were used.

#### 3.1 Natural damped frequencies

Numerical results for the damped natural frequency $\omega_{mn}$ for both a SS-SS-SS-SS and for a SS-C-SS-C plate were computed. These results are summarized in Table 1 and 2.

In Tables 1 and 2, we can observe that the damped natural frequency is proportional to values $(m, n)$. Moreover, for the SS-SS-SS-SS plate, the influence of the stiffness of the Winkler foundation is important at the first mode of the plate. However, for higher modes the effect of the stiffness of the Winkler foundation it not important. In addition, for the SS-C-SS-C plate, the influence of the stiffness of the elastic foundation is negligible.
Table 1: Values of the natural damped frequency $\omega_{mn}$ (rad/s) for a SS-SS-SS-SS plate for different values of stiffness (N/m$^2$)

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$K_f = 0$</th>
<th>$K_f = 3.90$</th>
<th>$K_f = 7.25$</th>
<th>$K_f = 9.90$</th>
<th>$K_f = 13.04$</th>
<th>$K_f = 30.38$</th>
<th>$K_f = 60.54$</th>
<th>$K_f = 118.58$</th>
<th>$K_f = 178.36$</th>
<th>$K_f = 240.10$</th>
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<tbody>
<tr>
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<td>1</td>
<td>3.15E+02</td>
<td>3.16E+02</td>
<td>3.16E+02</td>
<td>3.17E+02</td>
<td>3.17E+02</td>
<td>3.21E+02</td>
<td>3.26E+02</td>
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</tr>
<tr>
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<td>7.03E+02</td>
<td>7.04E+02</td>
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<td>7.06E+02</td>
<td>7.07E+02</td>
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<td>7.16E+02</td>
<td>7.22E+02</td>
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<td>3</td>
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<td>1.35E+03</td>
<td>1.36E+03</td>
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</tbody>
</table>

Table 2: Values of the natural damped frequency $\omega_{mn}$ (rad/s) for a SS-C-SS-C plate for different values of stiffness (N/m$^2$)

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
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</tbody>
</table>

3.2 Sound power level

We studied both the effect of the stiffness and the damping loss factor of the elastic foundation on the sound power level radiated by the plate. We consider that the plate is excited by an arbitrarily harmonic point force of amplitude $F = 9 \ N$, fixed at $(x = 0.02 \ m, y = 0.02 \ m)$ relative to the lower left corner. The sound power level (reference power of $10^{-12} \ W$) for a SS-SS-SS-SS is presented in Figure 1. As a reference, the value of the critical frequency $f_c$ has been.

It can be seen that the sound power level is independent of the stiffness of the elastic foundation for high values of damping loss factor of the foundation and the plate. However, for low values of damping loss factor of the foundation and plate, we notice that the sound power level of the plate has an inverse relationship with the stiffness of the elastic foundation.

4 Conclusions

An exact solution of the damped natural frequencies for both a SS-C-SS-C and SS-SS-SS-SS rectangular plate resting on a Winkler foundation has been presented. It is concluded that the damped natural frequencies depend upon the stiffness of the elastic foundation. However, the plate’s mode shapes do not depend upon the stiffness of the elastic foundation. Moreover, when the plate is subjected to forced vibration, we have proposed a matrix method to numerically...
estimate the effect of the elastic foundation on the sound radiation of the plate. The method has the advantage of not depend on the plate’s velocity distribution but is limited to frequencies in the mid and low range (discrete method). We observed that the sound power level of the plate has an inverse relationship with the stiffness of the elastic foundation. Thus, the results obtained showed that the use of an elastic foundation in a plate appears to be convenient for both structural noise and airborne noise control. It is expected that further work might be developed to estimate the sound radiation from circular or rectangular plates resting on a Winkler foundation with different boundary conditions. However, the proposed mathematical model should be validated experimentally.

![Figure 1: Sound power level for different values of $K_f$, $\eta_f$ and $\eta_p$.](image)

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