Impulse Representation of Sound Field due to a Rigid Wedge

Tjundewo Lawu, Mitsuhiro Ueda

Department of International Development Engineering
Tokyo Institute of Technology, Japan
lawu@ide.titech.ac.jp

Abstract

An impulse representation is described for calculating the diffraction wave due to a rigid wedge. The method is an approximation of the Biot-Tolstoy rigorous closed-form solution for diffraction of point source radiation by an infinite rigid wedge. Band-limited time-domain function can be reconstructed to the original waveform if it satisfies the sampling theorem, which assumes that sampling takes place at the lowest permissible sampling rate. Therefore, if the energy is concentrating between the first sampling intervals immediately after the rise time of the time-domain function, then the rigorous solution can be approximated as a delta function. This paper shows the description methods of the diffraction field near the ridge in 3-dimensional space. Using the proposed impulse representation, the numerical simulation was performed and the calculation accuracy was examined.

1. Introduction

Theoretical discussions of sound field by wedges generally are based on the vast literature describing the frequency dependence of continuous diffraction. On the other hand, the practical prediction of attenuation by wedges has leaned almost totally on laboratory experiments, which, of necessity, employ pulse techniques [1, 2]. The advantage of the pulse description when adapted to digital calculations is the superposition of identifiable diffracted contributions from each barrier edge can give the total impulse. The total temporal response can then be Fourier transformed to provide the desired frequency response of the rigid wedge.

There have been many reports published on the impulse response of diffraction by a rigid wedge or wedge-like object, and most of those relied on the rigorous impulse response of the wedge [3, 4]. A clear view of how the impulse response of diffraction arises from an arbitrary objects using virtual discontinuity principle of diffraction has also been reported [5].

In this paper, an impulse representation for calculating the diffraction wave due to a rigid wedge is described based on an approximation of the Biot-Tolstoy rigorous closed-form solution [3]. If a time-domain function of sound field \( p(t) \) is low-pass band limited to \( f_{\text{max}} \) cycles per second, then the signal can be reconstructed to its original waveform by an infinite number of equispaced sampling points, where the sampling rate is \( \varepsilon = 1/(2f_{\text{max}}) \) [6]. Accordingly, if most of the energy concentrates between the interval \( \pm \varepsilon/2 \) at \( t = t_0 \), then the time-domain function can be approximated as an impulse \( a\delta(t-t_0) \). In this case, the amplitude of the impulse \( a \) is expressed as

\[
a = \int_{t_0-\varepsilon/2}^{t_0+\varepsilon/2} p(t) \, dt \approx \int_{-\infty}^{+\infty} p(t) \, dt. \quad (1)
\]

A comprehensive treatment of the integration for diffraction field due to a rigid wedge is described in the subsequent section. Furthermore, the description methods of the diffraction field near the ridge in 3-dimensional space will also be discussed.

2. Diffraction Field due to a Rigid Wedge

The theoretical basis for the numerical procedure starts with a simple modification of the Biot-Tolstoy closed-form solution [3] for diffraction by an infinite rigid wedge. The expression is given in (2)-(5) [4] for the coordinates system shown in Fig. 1.

\[
p_d(t) = -c \left\{ \beta \right\} \exp \left( -\frac{\pi y}{\theta_W} \right) \frac{\sin \alpha}{4\pi \theta_W r_{O} r_{S} \sinh y}, \quad (2)
\]

where

\[
\beta = \frac{\sin \alpha}{1 - 2 \exp \left( -\frac{\pi y}{\theta_W} \right) \cos \alpha + \exp \left( -\frac{2\pi y}{\theta_W} \right)}, \quad (3)
\]

\[
y = \arccosh \frac{c^2 t^2 - (r_O^2 + r_S^2 + z_O^2)}{2r_O r_S}, \quad (4)
\]

\[
\alpha = \frac{\pi}{\theta_W} (\pi \pm \theta_O \pm \theta_S). \quad (5)
\]

Point source coordinates are \((r_O, \theta_O, z_O)\) and receiver coordinates are \((r_S, \theta_S, 0)\). The angle of the wedge in the free space is called \( \theta_W \). Note that the thin plate or semi-infinite plate is a wedge with \( \theta_W = 2\pi \). The term \((\pi \pm \theta_O \pm \theta_S)\) is written for simplicity, forming the four possible combinations of the angles \( \pm \theta_O \) and \( \pm \theta_S \).
in the integral, yields
\[ I(\varepsilon) = -\frac{1}{4\pi^2\epsilon_0} \int_{\varepsilon_k}^{1} \frac{\sin \alpha}{1 - 2\cos \alpha + \varepsilon^2} d\varepsilon, \quad (13) \]

where
\[ \exp \left( -\frac{\pi c}{\theta_W} \sqrt{\varepsilon_0} r_{OS} \right) = \varepsilon_x. \quad (14) \]

The limits of integration change from \( \tau = 0 \) to \( x = 1 \) and \( \tau = \infty \) to \( x = 0 \). Therefore, if \( \varepsilon_x \approx 0 \) can be approved, then \( I(\varepsilon) \approx I(\infty) \) and according to (1), the impulse representation is satisfied.

As an example, assuming that \( \varepsilon_x = 0.1 \), then the maximum observation distance becomes
\[ r_O < 0.19(\pi/\theta_W)^2\varepsilon_x. \quad (15) \]

In the case of right angle of wedge \( (\theta_W = 3\pi/2) \), \( c = 340 \text{ m/s,} \) and \( f_{\text{max}} = 40 \text{ kHz,} \) then the impulse representation is satisfied for \( r_O < 0.36 \text{ mm.} \)

If \( r_O \) satisfies the condition in (15) and let \( \varepsilon_x \approx 0 \), according to (1), the integration of (13) can be solved analytically [7], thus, \( p_d(t) \) can be approximated as
\[ p_d(t) \approx -\frac{1}{4\pi^2\epsilon_0} f(\alpha) \delta(t - \tau_0), \quad (17) \]

where \( f(\alpha) = [\pi - \text{mod} (\alpha, 2\pi)]/2 \) [8]. \text{mod} is the function to convert an angle into the value within 0 and 2\( \pi \).

Consider an observation point near the ridge of the wide-angle wedge \( (\pi < \theta_W < 2\pi) \). For the case of \( \theta_O = 0 \), the term within the curl brackets \( \{ \} \) in (17) is formed by \( \pi - \theta_S \) and \( \pi + \theta_S \), which are arised twice. Therefore, if \( \theta_S < \pi \), then the diffraction field near the ridge \( \theta_O = 0 \) becomes
\[ p_d(t) \approx \frac{1}{4\pi\epsilon_0} \left( \frac{2\pi}{\theta_W} - 2 \right) \delta(t - \tau_0), \quad (18) \]

and for the shadow boundary \( (\theta_S > \pi) \), the diffraction field becomes
\[ p_d(t) \approx \frac{1}{4\pi\epsilon_0} \left( \frac{2\pi}{\theta_W} \right) \delta(t - \tau_0). \quad (19) \]

At \( \theta_O = \theta_W \), the diffraction field can be found by similar approach.
3. Accuracy Tests

In order to evaluate the accuracy of the approximation by impulse representation, let us introduce an ultrasonic pulse as

\[
 f(t) = t \sin(\omega t) \\
 \left\{ \exp(-\alpha t) - \frac{\alpha}{(\alpha^2 + \omega^2)^2} \frac{(\beta^2 + \omega^2)^2}{\beta} \exp(-\beta t) \right\}, \tag{20}
\]

where \( \alpha \) and \( \beta \) were \( 6 \times 10^3 \) and \( 8 \times 10^2 \), respectively, and the center frequency of the pulse is 40 kHz. The signal observed by an observation point \( e(t) \) is then estimated by the convolution between Biot-Tolstoy solution and the ultrasonic pulse in (20).

For comparing the spreads of diffraction field estimated by impulse representation, variance ratio test was used, thus the impulse error can be written as

\[
 E = \frac{\sum_{i=0}^{n} |e(t) - e_i(t)|^2}{\sum_{i=0}^{n} |e(t)|^2}, \tag{21}
\]

where \( e_i(t) \) is the signal observed by the observation point calculated by the convolution between the impulse representation and the ultrasonic pulse.

3.1. Semi-infinite plate

To show the dependencies of diffraction field on the position of the observation point, let us first consider a semi-infinite plate \( (\theta_W = 2\pi) \). Figures 2(a) and (b) show the errors of diffraction field due to a semi-infinite plate as a function of \( \theta_O \) in the case of \( \theta_S = 30^\circ \) and \( \theta_S = 220^\circ \), respectively. The point source distance was 300 mm, and the distance of the observation point was 0.1\( \lambda \), 1\( \lambda \), and 10\( \lambda \). The accuracy of the impulse representation is pretty good near the boundary between illuminated and shadow regions of the source and its reflection. Figures 3(a) and (b) show the errors in 2-D polar plots for \( \theta_S = 30^\circ \) and \( \theta_S = 220^\circ \), respectively. \( x \) and \( y \) distances in the plots use logarithmic scale from 0.001\( \lambda \) to 10\( \lambda \) to show the error near the ridge of the semi-infinite plate. These plots also show that good approximation can be obtained for \( r_O \) less than 0.1\( \lambda \) along the ridge of the semi-infinite plate.

3.2. Rigid wedge

Figure 4 shows the errors of diffraction field due to a 315\(^\circ\) rigid wedge as a function of \( \theta_O \) in the case of \( \theta_S = 30^\circ \). The point source distance was 300 mm, and the distance

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**Figure 2:** The errors of diffraction field due to a semi-infinite plate \( (\theta_W = 2\pi) \). A point source was located at (a) \( r_S = 300 \text{ mm}, \theta_S = 30^\circ \) and (b) \( r_S = 300 \text{ mm}, \theta_S = 220^\circ \). Observation point was located at \( r_O = 0.1\lambda \) (solid line), \( r_O = 1\lambda \) (dotted line), and \( r_O = 10\lambda \) (dashed line).

**Figure 3:** Polar plots of diffraction error due to a semi-infinite plate \( (\theta_W = 2\pi) \). A point source was located at (a) \( r_S = 300 \text{ mm}, \theta_S = 30^\circ \) and (b) \( r_S = 300 \text{ mm}, \theta_S = 220^\circ \). \( x \) and \( y \) show the distances in logarithmic scale. The vertical line in the center of the plots shows the semi-infinite plate.
of the observation point was 0.1λ, 1λ, and 10λ. Again, as for the case of semi-infinite plate, the accuracy of the impulse representation is very good near the boundary between illuminated and shadow regions of the source and its reflection. Figure 5 shows the errors in 2-D polar plot for θS = 30°. This plot also shows that good approximation can be obtained for rO less than 0.1λ along the ridge of the rigid wedge.

As shown in the figure, the accuracy became worst within the region from θO = 180° to 210°. As shown in Fig. 4, at rO = 1λ, the error became worst when the observation angle was about 181°. To show this behavior, the time-domain response calculated by using Biot-Tolstoy closed-form solution is used. The polarity of the time response changes from positive to negative for θO = 180° to θO = 182°, and approaches zero at θO ≃ 181°. This means that \( \sum_{i=0}^{n} |e(t)|^2 \) in (21) becomes very small. Thus, the impulse representation error in (21) becomes large compared with a very small amount.

4. Conclusions

The impulse representation provides a significantly accurate solution to problems involving a rigid wedge near the boundary between illuminated and shadow regions also along the ridge of the rigid wedge. Noise barriers which have several diffracting edges will produce several impulses. By the described impulse representation, these impulses can be simply approximated by combining every wedge segment, therefore, it provides a logical basis for optimizing the design of complex barriers. Moreover, this impulse representation seems to be a potential approximation to solve the boundary value problems.

5. Acknowledgments

The authors would like to thank Yoshihiro Tomaru for many useful discussions.

6. References