Numerical Simulation of Wave Propagation in Complex Medium
Using with a Staggered Finite-Difference Method

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Abstract
In this paper we setup a model based on complex medium (composite), which considers randomly distributed characteristics of medium. A finite-difference method (FDM) of estimating velocity for composite based on elastic-wave theory is used.
The FDM wave equation using stress-particle velocity equations, and the scheme is second-order accurate in time and eight-order accurate in space. In order to improve the accuracy of the calculations, the staggered grids are generally used [1], [2] and [3].
This paper presents some examples of sound field simulation; the recorded forward full waveforms are processed by Fast Fourier Transform (FFT) and statistics method processing method to determine the P-wave velocities of composites [4].
The simulation results show there is a corresponding relationship between the P-wave velocity and the volume fraction (porosity) when the wavelength is greater than the size of the scatterers. When the scatterers impedance of media is smaller than that of the homogeneous media, the P-wave velocity has a regular decreasing and vice versa.

1. Introduction
In nature there are amount of medium which belong to complex random media, such as a layer of sedimentary rock in geology, porous rock, cell of biological tissue and so on.
The study of acoustic wave propagation in such complex medium is always an attracted topic and provides theoretical foundation for medical imaging, geophysics and NDT techniques.
Random medium is of complex interior structure, both the distribution and position of it’s composing cell are all fixed, it can be regarded as a sort of random medium which is made of by random scatters with different property, microstructure and size. The complexity of medium results in complex wave propagation phenomena such as reflection, diffraction and sharp discontinuous interface.

Recently such numerical analysis techniques as finite element method (FEM) and finite difference method (FDM) become popular in study of acoustic fields. In the FDM algorithm of this paper the wave velocity and stress are directly computed in discrete time steps and acoustic field phenomena can be animated [5].
This paper apply finite-deference scheme to modeling wave propagation in complex and random medium.
Numerical Simulation of sound field helps us to intuitively understand mechanisms of acoustic phenomena in the composite media; numerical analysis techniques as finite-difference method (FDM) become popular in acoustics recently [6].

2. Constitutive relation of elastic medium
In 2-D elastic media, the wave equation of motion to be implemented in a staggered-grid finite-difference is written as:

\[
\frac{\partial V_x}{\partial t} = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xz}}{\partial z} \quad (1a)
\]

\[
\frac{\partial V_z}{\partial t} = \frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zz}}{\partial z} \quad (1b)
\]

with the constitutive relation of

\[
\frac{\partial T_{xx}}{\partial t} = (\lambda + 2\mu) \left( \frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z} \right) - 2\mu \frac{\partial V_z}{\partial z} \quad (2a)
\]

\[
\frac{\partial T_{zx}}{\partial t} = (\lambda + 2\mu) \left( \frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z} \right) - 2\mu \frac{\partial V_x}{\partial x} \quad (2b)
\]

\[
\frac{\partial T_{zz}}{\partial t} = 2\mu \left( \frac{\partial V_z}{\partial z} + \frac{\partial V_x}{\partial x} \right) \quad (2c)
\]

where \(V_x\) and \(V_z\) are the particle velocity components in the \(x\)- and \(z\)- directions, respectively; \(T_{xx}\) and \(T_{zz}\) are...
the normal stress components in the x- and z- directions, and \( \tau_{xz} \) is the shear stress component in x-z plane.

Wave propagation in the above model is simulated with finite-difference method. In order to analyze the recorded waveforms frequency spectra for velocity estimation, the Fast Fourier Transform (FFT) and a statistical processing method are used to determine the P-wave velocities of composite media. By using the recorded waveforms at various source-receiver offsets, we are able to obtain P-wave velocity, and then we may establish the relationship between velocity and the fraction volume of random component.

In this paper an ideal model is considered based on random medium features, the model consists of one homogeneous medium and randomly distributed inclusions. The size of inclusions is assumed to be random normal distribution and its location is with uniformity distribution feature. The velocity and density contrasts between the host medium and the inclusions are white and black, respectively.

**Figure 1:** Waveforms of common source gather in x-direction of P waves in a homogeneous medium (a) and that in a random medium (b).

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**Figure 1** is one of the modeling results. Because of array sources almost cover one side of boundary and the wave propagation in this model propagates nearly a plane wave, which is shown in Figure 1a.

After the random scatters are added, the wave propagation becomes more complex than the homogeneous situation, and there are mass “coda waves” due to multi-scattering, which is shown in Figure 1b.

### 3. Implementation of Finite-difference Algorithm

In this paper the finite difference method (FDM) is used to simulate acoustic wave propagation in random medium. In general, the FDM arithmetic has follow advantages: (1) The acoustic source can be directly imported; and the value can be directly evaluated to displacement, velocity and stress at any location; (2) The flat free boundary is easily implemented by image method; (3) The order of time and space can be selected variously based on requirements of modeling; (4) The method is of simple expression of arithmetic; (5) By setting different receiver position, the acoustic field at any position can be modeled and displayed.

This paper implement center staggered grid method to set numerical region, the order of space is 8th and order of time is second.

There are mainly two ways in implementing the finite-difference algorithms for solving wave motion problems, i.e., the stress-particle-velocity equations and stress-displacement equations. In this paper, the stress-particle velocity equations were used

The merits of using these equations are that, when staggered grids are used, strong velocity-contrast interfaces can be handled with a high accuracy. This is because the derivatives of the medium density are not required in this algorithm.

In order to improve the accuracy of the calculations, the staggered grids are generally used. The locations of the stress and particle velocity components are shown in Figure 2.

The staggered grid for stress components, pressure, and particle-velocity components are located in the positions marked by \( C (T_{xx}, T_{zz}, \text{ and } P_f) \), \( A (V_x^s \text{ and } W_x) \), \( X (V_z^s \text{ and } W_z) \), and \( O (T_{xx}) \), respectively.
4. Numerical Examples

This paper considers a numerical model which constituted with two kind of solid medium. The calculation includes two cases: the condition of case 1 is that the inclusions with low impedance are distributed in a host medium with high impedance; and the condition of case 2 is that the inclusions with high impedance are distributed in a host medium with low impedance. The parameters of modeling and properties of medium are shown as table 1 and table 2 respectively.

Table 1: Parameters of FDM simulation

<table>
<thead>
<tr>
<th>Total space steps (x,y)</th>
<th>Total time steps</th>
<th>Space step (x,y)</th>
<th>Time step</th>
<th>Diameter of inclusions</th>
</tr>
</thead>
<tbody>
<tr>
<td>800</td>
<td>6000</td>
<td>0.004 m</td>
<td>0.5 µs</td>
<td>0.08 m</td>
</tr>
</tbody>
</table>

Table 2: Parameters of elastic medium

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Medium 1</th>
<th>Medium 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>P-wave velocity (m/s)</td>
<td>3600.</td>
<td>1800.</td>
</tr>
<tr>
<td>S-wave velocity (m/s)</td>
<td>2079.</td>
<td>1040.</td>
</tr>
<tr>
<td>Density (kg/m³)</td>
<td>2600.</td>
<td>2400.</td>
</tr>
</tbody>
</table>

The pulse of source is can be referenced as below Figure 3. The y-axis is amplitude and the x-axis is travel time.

The centre frequency of the source is 10 KHz. The source wavelet function is the first derivative of the Gaussian function, and the highest frequency is four times the centre frequency. The waveform is perfectly symmetric about the middle that crosses zero value.

The result of modeling (as shown Fig 4) shows that variation of waveform about homogeneous medium (a) and random medium (b). The y-axis is amplitude and the x-axis is travel time.

The random inclusions affect the received waveform and made it much more complex. In the Figure 4(a), P-wave and S-wave are all clear but the P-wave and S-wave propagation have a different ways in the Figure 4(b). It became very difficulty to obtained the first arrive.
Figure 5 shows results of changing of velocity with volume fraction (porosity). The y-axis is P-wave velocity and the x-axis is volume fraction.

5. Conclusions

In this paper, some examples of acoustic field based on the numerical calculation using the FDM have been presented. The velocity of wave propagation in composite is analyzed weightly. The finite difference is an effective method which can be applied to simulation wave propagation in model of composite; some quantified results which are related to microstructure are gained. The comparison of modeling results between average time model and random model shows random model is a reasonable model compared with the realistic composite.

6. Reference