Non linear wave propagation in cylindrical air-filled tubes.

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Abstract: From three dimensional conservation laws, quasi-unidimensional generalised Burgers equations, taking into account both non linear phenomena and viscothermal losses, are determined for cylindrical air-filled tubes. The validity of the different approximations is discussed. A numerical solution of these equations is compared with preliminary experimental results.

INTRODUCTION

The plane wave deformation during propagation in an acoustic wave guide is mainly controlled by the viscothermal losses in the wall boundary layer (Kirschho 1868 [1]) and by the non-linear effects in high sound level (Chester 1964 [2]). These locally negligible effects are cumulative along the propagation. Multiple Scale Methods applied to the basic equations lead to generalised Burgers equations in the progressive wave hypothesis (Sugimoto 1991 [3]). Two generalised Burgers equations are obtained from the basic equations in the non progressive wave hypothesis. The non-linear interaction between the incoming and the outgoing waves is non cumulative. A solution of these equations from two boundary conditions is presented and compared with experimental results.

FROM BASIC EQUATIONS TO UNIDIMENSIONAL NON-LINEAR EQUATIONS

The fluid is considered as Newtonian, in laminar regime, in quiescent state, according to the equation of state of a perfect gas. The following calculations are based on the existence of three non dimensional coefficients which are much smaller than unity:

\[ M = \frac{\hat{u}}{c_0} \ll 1, \quad \frac{1}{Re} = \frac{\mu_0}{\rho_0 c_0^2} \ll 1, \quad Sh = \frac{\xi}{R} = \frac{\mu}{\rho_0 \omega R} \ll 1 \]

where \( M \) is the Mach number depending on the nonlinearities, \( Re \) the acoustic Reynolds number depending on the volumic losses, \( Sh \) the shear number depending on the boundary layer losses. \( \hat{u}, c_0, \rho_0, \mu, R \) are the acoustic velocity, the speed of sound in the ambient gas, the atmospheric density, the shear viscosity and the radius of the tube. Typical experimental conditions (frequency range 50-500 Hz, radius around 1 cm, level range 140-175 dB) are corresponding to \( Sh = 10^{-2}, 1/Re = 10^{-7}, M = 10^{-2} \) (1/Re is at least three orders of magnitude smaller than \( M \) and \( Sh \)).

The three conservation laws (mass, momentum and energy) are written with the cylindrical coordinates, because of the axial symmetry of the tube. The set of non dimensional equations are simplified by comparing the order of each term. Each order of magnitude is defined by non dimensional numbers which are functions of the former coefficients \( M, Sh, 1/Re \).

Before simplifying equations, the acoustic variables must be expressed as non dimensional variables \( P, \rho, T \) and \( u \), and scaled so that their order of magnitude is unity:

\[ \hat{P} = P_0 (1 + MP), \quad \hat{\rho} = \rho_0 (1 + Mp), \quad \hat{T} = T_0 (1 + MT), \quad \hat{u} = c_0 Mu, \]

where \( P_0 \) and \( T_0 \) are the ambient pressure and ambient temperature.

Starting from linear acoustic results, the order of magnitude of the derivative variables are estimated in the main field and in the boundary layer. These two parts of the tube are studied separately, then matched with the radial velocity at the interface.

Non-linear one dimensional propagation equations (closed Chester 1964 [2]) are obtained:

\[ \left\{ \frac{\partial}{\partial t} + (Mu + c) \frac{\partial}{\partial x} \right\} \left( Mu \pm \frac{2c}{\gamma - 1} \right) = \pm 2MSh \left( 1 + \frac{\gamma - 1}{\sqrt{Pr}} \right) \frac{\partial u}{\partial x} \pm \frac{1}{\sqrt{\pi t}} + O \left( M^2 Sh, MSh^2, \frac{M}{Re Sh} \right), \quad (1) \]
where \( c = \left(1 + M_p\right)^{1/2} \) (*is the convolution product; \( Pr \) denotes the Prandtl number).

The order of magnitude of the first neglected terms are \( MSh \) (boundary layer non-linearities), \( Sh^2 \) (geometry of the tube), \( 1/(ReSh) \) (plane wave approximation applied to the acoustic pressure). Notice that the viscosity effect in the mainstream flow is neglected since \( 1/Re \) is negligible compared to \( MSh, Sh^2 \) and \( 1/ReSh \).

**MULTIPLE SCALE METHOD (MSM) AND GENERALISED BURGERS EQUATIONS**

A "slow" axial scale is defined: \( X = Mx \). Then the MSM is based on three non dimensional variables \( \sigma, \theta^+ \) and \( \theta^- \) defined below. There are non-linear interactions between the incoming and outgoing waves, but these interactions are not cumulative (it remains a local effect) and so will be neglected. The two following non-linear differential Burgers equations are obtained:

\[
\frac{\partial q^\pm}{\partial \sigma} + q^\pm \frac{\partial q^\pm}{\partial \theta^\pm} = \frac{T}{\varepsilon} \frac{\partial q^\pm}{\partial \theta^\pm} \ast \frac{1}{\sqrt{\pi \theta^\pm}}
\]

with \( \sigma = \frac{y-1}{2} X \); \( T = \text{Sh} \left(1 + \frac{y-1}{Pr}\right) \), \( \varepsilon = \frac{y-1}{2} M \); \( \theta^\pm = \theta \pm \xi \).

\( q^\pm \) is the incoming wave (+), or outgoing wave (-) \( (u = q^+ - q^-) \). \( \gamma \) denotes the ratio of the specific heats.

**NUMERICAL SOLUTIONS AND PRELIMINARY EXPERIMENTAL RESULTS**

The equations (2) are solved between the input (E), and output (S) of the cylindrical tube using two boundary conditions: the acoustic pressure at (E) and pressure or impedance at (S). Everywhere the acoustic signal is considered as the sum of the incoming \( p^+ \) and the outgoing \( p^- \) waves. These two periodic waves are assumed to be a Fourier sum of \( N \) harmonics. The iterative method is based on a numerical convergence (Newton-Raphson method) of the components of \( p^+ \) towards the solution of a set of non-linear equations. At each loop, \( p^+ \) is propagated from (E) to (S) using a finite different approximation of the first equation of (2). Using the known value of the acoustic pressure at (S), \( p^- \) is deduced and propagated from (S) to (E) by applying second equation of (2). Finally \( p^+ \) is induced in (E) from the known value of the acoustic pressure at this point. The comparison between the components of \( p^+ \) at the beginning and at the end of the loop gives the non-linear equations set (harmonic balance method).

Preliminary experiments have been carried out with P. Durrieux in the University of Eindhoven (Menguy 1997 [4]): the experimental set-up consists of a ten meter tube impregnated with pressure transducers along them. The source was a siren. The two extreme transducers are placed 6.73 meters apart; these transducers define the positions (E) and (S). The measurements from the two extreme transducers set the boundary conditions which are applied to the numerical method. Different levels from 140 dB up to 175 dB were tested, at frequencies around 100 Hz. The numerical method was used to calculate the acoustic pressure at each of the intermediate transducer positions. There is good agreement between the experimental and numerical results: at each intermediate transducer position there is less than 1% discrepancy in the amplitude of the fundamental component. Notice that the non-linear and linear theories differ up to 10%.

**REFERENCES**


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