Acoustical Helicoidal Waves and Laguerre-Gaussian Beams: Applications to Scattering and to Angular Momentum Transport

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Abstract: Some properties of acoustical traveling waves with helicoidal or "spiral-like" wavefronts are analyzed. These include orbital angular momentum flux and potentially useful imaging or scattering properties for the discrimination of axisymmetric objects from other objects. Measurements are summarized for a simple single-element ultrasonic transducer for generating approximately helicoidal beam-waves in water.

INTRODUCTION

A potentially useful class of exact and approximate solutions of the Helmholtz equation that do not appear to have been well developed in acoustics are traveling waves having helicoidal wavefronts. The wavefields are cylindrically symmetric except for an azimuthal angular dependence of \( \exp(i\theta) \) where \( \theta \) is the azimuthal angle. The amplitude vanishes at a screw-phase dislocation on the z propagation axis. Examples of paraxial helicoidal waves are the Laguerre-Gaussian (LG) beam solutions of the parabolic wave equation which are well known in optics. Linearly polarized electromagnetic LG beams transport angular momentum about the z axis and have been used to apply a torque to optically absorbing spheres, which gives an alternative to torque application based on circular polarization. Since the angular momentum transport is the result of the helicoidal wavefront geometry, the corresponding acoustical LG beam carries an angular momentum analyzed here. Using cylindrical coordinates \((r, \theta, z)\) and the standard optical result, the pressure may be written as the real part of

\[
p(r, \theta, z) = A_{mn} \left[ 1 + \left( \frac{z}{z_R} \right)^2 \right]^{-1/2} e^{-i(w/2)^2} e^{-i\psi(z/z_R)} e^{i\omega z} e^{i\tilde{r} (r/w)^2} L_n^m \left( \frac{2r^2}{w^2} \right),
\]

where \( w = \sqrt{(z^2 + z^*^2)/kz} \), \( z^* \) is a complex conjugate, \( A_{mn} \) is a constant, \( L_n^m \) is the associated Laguerre polynomial of the indicated argument, \( z_R = kw^2/2 \) is the Rayleigh range, \( w \) is a beam waist parameter, \( R = (z_R^2 + z^2)/z \), \( \psi = (m + 1 + 2n) \tan^{-1}(z/z_R) \) is a generalized Guoy phase shift, \( m = 0, 1, 2, ... \) affects the number of radial nodes, and \( z \) is the distance from the beam waist. The helicoidal wavefront is a consequence of the \( \exp(i\theta) \) factor and in the special case of \( m = 0 \), \( R(z) \) take on the significance of the phase-front radius of curvature.

RATIO OF AXIAL ANGULAR MOMENTUM FLUX TO BEAM POWER

Let \( \varphi = (ip/\omega p) \) denote the complex velocity potential and \( v = -\nabla \text{Re}[\varphi] \) be the fluid velocity where \( \text{Re} \) denotes the real part. From Eq. (1), the azimuthal velocity is \( v_\theta = \text{Re}[-i\varphi r] \). The average axial angular momentum density of the beam is \( \langle (\delta\rho)rv_\theta \rangle \) where \( \delta\rho = (c^{-2}) \text{Re}[p] \) is the first-order change in density due to the acoustic wave and \( \langle \rangle \) denotes a time average. The angular momentum flux \( \langle L_z \rangle \) and power \( P \) of the beam are

\[
\langle L_z \rangle = 2\pi \int_0^\infty \langle (\delta\rho)rv_\theta \rangle rdr, \quad P = 2\pi \int_0^\infty \langle \text{Re}[-ik\varphi] \text{Re}[p] \rangle rdr,
\]

where \( P \) is the integral of the local average acoustic intensity \( \langle \text{Re}[p] \rangle \). Inspection of Eq. (2) gives \( \langle L_z \rangle / P = m/\omega \) for each value of \( m \) and \( n \) in Eq. (1). This ratio is the same as for an electromagnetic beam. Consequently absorption of acoustic energy from beams with \( m \neq 0 \) will produce an axial torque on the absorber.

SYMMETRY PROPERTIES AND BACKSCATTERING

For the situations of interest, \( m \neq 0 \) and \( p \) vanishes on the z axis since the \( L_n^m \) are regular; for example \( m = 1 \), \( n = 0 \) gives \( L_1^0 = 1 \). The symmetry properties of a scatterer with respect to the z axis may be revealed by detecting the backscattering. Suppose, for example, the target is an axisymmetric reflector as in the case of a perpendicular mirror or a spherical reflector located on the axis. For an axisymmetric receiver transducer with a transfer function containing a unimodular phase factor \( s(\theta) \) the complex output voltage contains of factor proportional to
where the $\exp(\text{i}m\theta)$ is due to the illumination and the target symmetry. $V$ vanishes if either $s$ is constant (as in a spatially flat transducer) or if the source transducer is used as a receiver so that from reciprocity $s = \exp(\text{i}m\theta)$. If $s$ is set to detect a wave of opposite pitch as the transmitted wave, $s = \exp(-\text{i}m\theta)$ and $V$ is maximized.

**HELICOIDAL TRANSDUCER**

The transducer used to generate acoustical helicoidal beams is shown in Fig. 1. It is composed of an annular sheet of PVDF attached at its outer edge to a ring of marine brass. The active portion of the PVDF has an outer radius of $a = 4.5$ cm and an inner radius of $b = 1.9$ cm. The ring was cut at a point near the transducer's support such that the ring could be twisted like the coil of a spring. By placing a cut through the PVDF corresponding to the break in the ring, the ring can be deformed such that the height of the surface of the transducer can be described by $z_T = \frac{\lambda}{2\pi} \theta$, where $\lambda$ is the wavelength of the beam. This should produce a phase-front which has a $\theta$-dependence similar to that of a LG beam with $m=1$.

The transducer was suspended in a 8' x 8' tank of water and driven at a frequency of 300 kHz. Using a positioning system, it was possible to sample the acoustical signal from the transducer in the plane perpendicular to the face of the transducer at $z = 74.6$ cm. The intensity distribution in this plane with samples taken using an Edosphere hydrophone at 0.5 cm increments is plotted in Fig. 2. In the center of the beam is an approximate null which would be expected as the phase becomes indeterminate at that point. This can also be seen by examining the phase of the beam directly as in Fig. 3(b). The phase of a LG beam with a beam waist, $w_0$, equal to the outer diameter of the transducer is shown in Fig. 3(a) for comparison. Although the beam generated by the transducer does not have the same uniform structure, it does have a similar spiral in the phase distribution. The intensity distribution in Fig. 2 also displays strong similarities to the intensity distribution of the LG beam, although it is not shown here. These similarities indicate that this may be a useful transducer design for the generation and reception of helicoidal waves.

![FIGURE 1. Transducer](image1)

![FIGURE 2. Intensity distribution of 300kHz beam at $z = 74.6$ cm. The central dark spot is low intensity.](image2)

![FIGURE 3. (a) Phase distribution of a 300kHz LG beam at $z = 74.6$ cm and $w_0 = 4.5$ cm (b) Phase distribution for beam from transducer. Dark-to-light indicates a phase increment of 360°.](image3)

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**REFERENCES**