The Form of the Normal Mode that Ensures Escape with Certainty from a Surface Channel

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Abstract: A normal mode is synthesized so that it can escape from a surface channel with certainty and without reflection.

INTRODUCTION

Surface channels trap those rays that are reflected internally before reaching a deep sound channel according to contemporary ray tracing. The wave representation describes some leakage of trapped normal modes to the deep sound channel. Recent advances in quantum mechanics (1) have shown that a wave function having sub-barrier energy can tunnel through a barrier with certainty and without reflection. Herein, a synthesized normal mode is shown to escape from a surface channel with certainty and without reflection. These synthesized normal modes, which may be unfamiliar to many workers, are solutions to the wave equation. This investigation considers a bottomless sound velocity profile, \( C(z) \), whose inverse square, \( C^{-2}(z) \), is bilinear and forms a surface channel. The findings are generalized to include various surface and deep sound channels. While an investigation of quantum tunneling with certainty was inspired by and initially solved in a trajectory representation (1), which is analogous to rigorous ray tracing (2), we have used a wave representation herein because workers are more familiar with normal modes. Still, rigorous ray tracing is more intuitive for an analysis of certain escape.

DEVELOPMENT OF THE SYNTHESIZED NORMAL MODE

Let us consider a surface channel formed by a sound velocity profile, \( C(z) \), whose inverse square, \( C^{-2}(z) \), is given in units of \((\text{sec/m})^2\) by

\[
C^{-2} = \begin{cases} 
1400^{-2} + \alpha(z - q), & z \leq q \\
1400^{-2} + \beta(z - q), & z > q
\end{cases}
\]

where \( \alpha, \beta > 0 \). As \( C(z) \) is dependent upon depth only, we can reduce the wave equation by separation of variables to the Helmholtz equation

\[
\phi_{zz} + \omega^2(C^{-2} - C_m^{-2})\phi = 0
\]

where \( C_m \) is a separation constant. If \( C_m < C(q) \), then contemporary thought considers that the normal mode for a source in the surface channel is a “trapped mode” albeit with some leakage. We shall investigate this more carefully. We examine what normal modes can be constructed if \( C_m^{-2} = 1400^{-2} + \lambda \) where \( 0 < \lambda < \alpha q \).

The solutions to Eq. (2) are Airy functions. In the surface channel, \( z < q \), the independent solution pair is \( \text{Ai}[\omega^{2/3} \alpha^{1/3}(z - q + \lambda/\alpha)] \) and \( \text{Bi}[\omega^{2/3} \alpha^{1/3}(z - q + \lambda/\alpha)] \). In the deep channel, \( z > q \), the independent solution pair is \( \text{Ai}[\omega^{2/3} \beta^{1/3}(q - z + \lambda/\beta)] \) and \( \text{Bi}[\omega^{2/3} \beta^{1/3}(q - z + \lambda/\beta)] \).

Let us lay the foundation for constructing a normal mode that will escape from the surface channel with certainty. This mode penetrates without any reflection through the depth domain, \( q - \lambda/\alpha < z < q + \lambda/\beta \), where Snell’s law forbids contemporary ray tracing.

Let us generate a transmitted wave that will look like a running wave deep in the deep zone. We let \( \zeta = \omega^{2/3}\beta^{1/3}(q - z + \lambda/\beta) \) for convenience. We want a transmitted wave, \( \phi_\beta \) in the deep channel given by

\[
\phi_\beta = \text{Ai}(\zeta) + i \text{Bi}(\zeta), \quad \zeta > -\omega^{2/3}/\beta^{2/3} \lambda \text{ or } z > q
\]

\[
= [\text{Ai}^2(\zeta) + \text{Bi}^2(\zeta)]^{1/2} \exp \left[ i \arctan \left( \frac{\text{Bi}(\zeta)}{\text{Ai}(\zeta)} \right) \right], \quad \zeta > -\omega^{2/3}/\beta^{2/3} \lambda \text{ or } z > q
\]

\[
\approx \pi^{-1/2} \zeta^{-1/4} \exp \left[ -i(2/3) \zeta^{3/2} - \pi/4 \right], \quad \zeta \gg 0 \text{ or } z \gg q + \lambda/\beta.
\]
Equation (3) describes a running wave deep in the deep channel. We now determine the structure of the wave, $\phi_\succ$, in the surface channel, $z < q$, that corresponds to $\phi_\prec$. We let $\xi = \omega^{2/3} \alpha^{1/3} (z - q + \lambda/\alpha)$ for convenience. In the surface channel, our wave is $\phi_\prec = A \text{Ai}(\xi) + B \text{Bi}(\xi)$. The coefficients, $A$ and $B$, are determined by continuity of the logarithmic derivative of $\phi$ at $q$ where

$$\phi_\succ(q) = \text{Ai}[(\omega/\beta)^{2/3} \lambda] + i \text{Bi}[(\omega/\beta)^{2/3} \lambda]$$

and

$$\phi_{\succ\prec}(q) = \{A'[\omega/(\beta)^{2/3} \lambda] + i B'[\omega/(\beta)^{2/3} \lambda]\} \omega^{2/3} \beta^{2/3}.$$ (5)

After the usual, straightforward, but tedious and unilluminating matching of the logarithmic derivatives of $\phi_<$ and $\phi_\succ$ at $z = q$, we find that

$$A = \pi \{\phi_\succ(q) B'[\omega/(\alpha)^{2/3} \lambda] - \omega^{-2/3} \alpha^{-1/3} \phi_{\succ\prec}(q) B[(\omega/\alpha)^{2/3} \lambda]\}$$

and

$$B = \pi \{\omega^{-2/3} \alpha^{-1/3} \phi_{\succ\prec}(q) A[\omega/(\alpha)^{2/3} \lambda] - \phi_\succ(q) A'[\omega/(\alpha)^{2/3} \lambda]\}$$

By Eqs. (4) (7), $A$ and $B$ are complex. Thus, we have in the surface channel that

$$\phi_\prec(\xi) = A \text{Ai}(\xi) + B \text{Bi}(\xi) = \frac{\{\Re[A] \text{Ai}(\xi) + \Re[B] \text{Bi}(\xi)\} + i \{\Im[A] \text{Ai}(\xi) + \Im[B] \text{Bi}(\xi)\}}{\text{Ci}(\xi)} \frac{1}{\text{Di}(\xi)}$$

where $\text{Ci}(\xi)$ and $\text{Di}(\xi)$ are defined by Eq. (8) to form an alternative pair of independent solutions to Eq. (2) in the surface channel by the superposition principle.

We now synthesize a running wave for $\phi_\prec(\xi)$ from $\text{Ci}(\xi)$ and $\text{Di}(\xi)$ by

$$\phi_\prec(q) = \text{Ci}(\xi) + i \text{Di}(\xi) = [\text{Ci}^2(\xi) + \text{Di}^2(\xi)]^{1/2} \exp \left[ i \arctan \left( \frac{\text{Di}(\xi)}{\text{Ci}(\xi)} \right) \right].$$

This running wave, $\phi_\prec(\xi)$, is a normal mode of Eq. (2) in the surface channel. The phase of $\phi_\prec(\xi)$ increases monotonically with depth. This synthesized normal mode escapes the surface channel with certainty and without reflection. Our results are general as $q$, $\alpha$, $\beta$ and $\lambda$ are parameters. Our results can be extended to other surface channels for which $C^2(z)$ is not bilinear.

Any concern that Eq. (9) represents a clever disguise of an incident and a reflected wave, $\text{Ai}(\xi)$ and $\text{Bi}(\xi)$ respectively, has already been put to rest elsewhere (1) where it was shown by the reversibility of the mappings of the superposition principle that the pair $(\text{Ai}, \text{Bi})$ can be constructed from the alternative pair $(\text{Ci}, \text{Di})$. No particular set of independent solutions is favored.

REFERENCES