New Symmetries and Conservation Laws for Lossless KZK Equation

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Abstract: Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation is widely used to describe intense acoustic beams. No general analytical solution has been obtained for this nonlinear equation, so numerical methods are usually employed. Known analytical approaches are associated with various approximations. Additional powerful mathematical tool is group analysis that enables to find general symmetrical properties of differential equations. These symmetries help to generalize known analytical and numerical solutions, to derive new solutions, to obtain conservation laws. In this work, results of group analysis of KZK equation are presented. Classical Lie symmetries (geometric symmetries) are found for the KZK equation in cases of quadratic and cubic nonlinearities. Besides, results of group classification of lossless KZK equation, usually called KZ equation, with arbitrary nonlinear term are presented. It is shown that KZ equation can be written in Euler-Lagrange form. All the geometric symmetries are found and some classes of Lie-Backlund symmetries are considered. The largest number of the symmetries corresponds to the cases of quadratic and cubic nonlinearities. Besides scaling and translation groups of symmetries, there exist additional transformation groups which are not evident from the physical point of view. Examples of obtaining of new solutions and deriving of reduced equations are presented. New conservation laws for intense acoustic beams are obtained using Noether theorem.

INTRODUCTION

Intense acoustic beams are governed by Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation,
\[
\frac{\partial}{\partial \tau} \left[ \frac{\partial u}{\partial z} + P(u) \frac{\partial u}{\partial \tau} - A \frac{\partial^2 u}{\partial \tau^2} \right] - \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{1}
\]
here all variables are dimensionless, \( u \) represents waveform, \( \tau \) is retarded time, \( z \) is axial coordinate, \( x \) and \( y \) are transversal coordinates, \( A \) is absorption parameter. Function \( P(u) \) accounts for nonlinearity of the medium. In case of acoustic waves propagating in liquid, the nonlinearity is usually considered as a quadratic one \( P(u) = u \). However, other types of the nonlinearity may be of interest. No general analytical solution has been obtained for Eq.(1), so numerical methods are usually employed. Additional powerful mathematical tool is group analysis that enables to find general symmetrical properties of differential equations. Consider the following general form of transformation of the variables:
\[
\tau = T(\tau, x, y, z, u; \lambda), \quad \tau = X(\tau, x, y, z, u; \lambda), \quad \tau = Y(\tau, x, y, z, u; \lambda), \quad \tau = Z(\tau, x, y, z, u; \lambda), \quad \tau = U(\tau, x, y, z, u; \lambda), \tag{2}
\]
where functions \( T, X, Y, Z, \) and \( U \) describe the transformation, \( \lambda \) is its parameter. The reversible transformation given by Eq.(2) is called Lie symmetry (geometric symmetry) of differential equation, if the result of the transformation of any solution \( u(\tau, x, y, z) \) is again a solution \( u(\tau, x, y, z) \) of this equation. There exists a regular mathematical technique which enables to calculate Lie symmetries of differential equations [1-3]. Despite the KZK equation is well-accepted equation of nonlinear acoustics, its symmetries have not been adequately studied.

SYMMETRIES OF KZK EQUATION FOR QUADRATIC AND CUBIC NONLINEARITIES

We calculated all possible Lie symmetries for Eq.(1) in the cases \( P(u)=u \) and \( P(u)=u^2 \). In both cases the total number of symmetries is 8. All of them have clear physical sense: 1 symmetry is associated with scaling transformation, 4 symmetries correspond to translations of the variables \( \tau, x, y, z \), and the rest 3 symmetries can be interpreted as invariance of Eq.(1) with respect to rotations in the planes \((x,y),(\tau,x),(\tau,y)\).
One of the important applications of the symmetries is a generalization of known (analytically or numerically) solution. The commonly accepted way of such a generalization is a consistent stretching of the variables. However, the symmetries provide less trivial results [4]. Consider acoustic beam governed by solution 
\( u = F(\tau, x, y, z) \). One of the symmetries of Eq.(1) produces the whole family of new solutions \( u = F(\tau + \lambda x, x + \lambda y, y, z) \), which describes acoustic beam inclined at angle \( \lambda \) to the axis 0z. For quadratically nonlinear medium more general result follows from the symmetries: 
\( u = xq'x/2 - qq'/4 + F(\tau - xq'/2 + qq'/4, x - q, y, z) \), where \( q = q(z) \) is arbitrary function of axial coordinate. This expression shows how the structure of the beam is distorted when it interferes with inhomogeneous field \( u = xq'/2 - qq'/4 \). Note that the beam axis becomes curved, as it is shifted in the \( x \)-direction by distance \( q(z) \).

GROUP CLASSIFICATION OF LOSSLESS KZK EQUATION WITH ARBITRARY NONLINEAR TERM

Consider the Eq.(1) without dissipative term (\( A = 0 \)). This (lossless) KZK equation governs nonlinear evolution of acoustic wave in ideal medium until the waveform becomes shocked. We studied the symmetrical properties of this equation on the assumption that \( P(u) \) is arbitrary smooth function. Group analysis of the differential equation containing arbitrary parameters or functions is usually referred to as group classification, it is more complicated than the group analysis of some particular equation. The results of group classification of Eq.(1) are as follows. There are 8 symmetries which do not depend on the function \( P(u) \). These are a scaling symmetry, 4 translations of the variables, and 3 rotations. Another scaling symmetry exists for \( P(u) = \exp(u), P(u) = \ln(u), \) and \( P(u) = u^a, \) where \( a \) is nonzero constant. All other symmetries are associated only with particular cases \( P(u) = 0, \) \( P(u) = u, \) and \( P(u) = u^2 : \) in addition to the mentioned symmetries, in the linear case there are 9 symmetries, for quadratically nonlinearity 4 extra symmetries exist, and 1 extra symmetry appears in case of cubic nonlinearity [5]. Again, the symmetries help to generalize known solutions. An interesting example describing intense beam focusing in cubically nonlinear medium was previously reported in the paper [6].

We showed that the lossless KZK equation can be written in Euler-Lagrange form. According to Noether theorem, each symmetry, which does not change Lagrange function, is associated with some conservation law: 
\( \partial J_x / \partial \tau + \partial J_y / \partial x + \partial J_z / \partial y + \partial J_z / \partial z = 0 \). The values \( J_x, J_y, J_z \) were calculated for all symmetries. Note that the conservation law means that integral \( I = \int J_x d\tau dxdy \) is not changing with the propagation distance: \( dI/dz = 0 \). Here is one of the integrals in case of quadratic nonlinearity medium:

\[ I = c_1z + c_2z^2 + \int \left[ \tau^2 u - \frac{3}{4} (x^2 + y^2)u^2 \right] d\tau dxdy , \]

where \( c_1 \) and \( c_2 \) are known constants. Some other examples can be found in Refs. [5,6]. The conservation laws can be used to control numerical algorithms.

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REFERENCES